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Subconstituents of symplectic graphs[☆]

Feng-gao Li[a], Yang-xian Wang[b]

[a] Department of Mathematics, Hunan Institute of Science and Technology, Yueyang, Hunan 414006, PR China
[b] Department of Mathematics, Hebei Normal University, Shijiazhuang, Hebei 050016, PR China

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Abstract

We show that the subconstituents of the symplectic graph $Sp(2\nu, q)$ are strictly Deza graphs except the trivial case when $\nu = 2$. The chromatic numbers of those subconstituents are also given.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements and $\mathbb{F}_q^{(2\nu)}$ the 2ν-dimensional row vector space over $\mathbb{F}_q$, where q is a prime power and $\nu \geq 1$ is an integer. For any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q^{(2\nu)}$, denote by $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ the subspace of $\mathbb{F}_q^{(2\nu)}$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$. When $\alpha = (a_1, a_2, \ldots, a_{2\nu}) \in \mathbb{F}_q^{(2\nu)}$ we also write $[\alpha] = [a_1, a_2, \ldots, a_{2\nu}]$ for simplicity. For $1 \leq i \leq 2\nu$, we use e_i to denote the 2ν-dimensional row vector whose i-th entry is 1 and all other entries are zero. Denote by ${}^tA$ the transpose of the matrix A.

Let

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}.$$

Then the set $\left\{T \in M_{2\nu}(\mathbb{F}_q) \mid TK{}^tT = K\right\}$ forms a group under matrix multiplication, called the *symplectic group* of degree 2ν with respect to K over $\mathbb{F}_q$, and is denoted by $Sp_{2\nu}(\mathbb{F}_q)$.

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E-mail address: fenggaol@163.com (F.-g. Li).

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The *symplectic graph* $Sp(2\nu, q)$ relative to K over $\mathbb{F}_q$ is the graph with the set of one dimensional subspaces of $\mathbb{F}_q^{(2\nu)}$ as its vertex set and the adjacency defined by

$$[\alpha] \sim [\beta] \quad \text{if and only if} \quad \alpha K^t \beta \neq 0 \quad \text{for 1-dimensional subspaces } [\alpha], [\beta].$$

Note that $Sp(2\nu, q)$ is a clique with $q + 1$ vertices when $\nu = 1$, we assume $\nu \geq 2$ in the rest of this paper. Denote by Γ the symplectic graph $Sp(2\nu, q)$ for simplicity. The vertex set of Γ is written by $V(\Gamma)$, and dist$([\alpha], [\beta])$ means the distance between vertices $[\alpha]$ and $[\beta]$. Note that the diameter of $Sp(2\nu, q)$ is 2 when $\nu \geq 2$. For any $[\alpha] \in V(\Gamma)$, we use $\Gamma_i([\alpha])$ to denote the set of vertices $[\beta]$ in $Sp(2\nu, q)$ satisfying dist$([\alpha], [\beta]) = i$, where $i = 1, 2$.

A connected simple graph G is called *strongly regular* with parameters (v, k, λ, μ) if it consists of v vertices such that for any $x, y \in V(G)$,

$$|G(x) \cap G(y)| = \begin{cases} k & \text{if } x = y, \\ \lambda & \text{if } x, y \text{ are adjacent}, \\ \mu & \text{otherwise}, \end{cases}$$

where $G(x)$ is the set of neighbors of x. A graph G is said to be *n-partite* if there are subsets $X_1, X_2, \ldots, X_n$ of the vertex set $V(G)$ such that $V(G) = X_1 \cup X_2 \cup \cdots \cup X_n$, where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and that there is no edge of G joining two vertices of the same subset. The *chromatic number* $\chi(G)$ of G is defined as being the minimal n such that G is n-partite. It is known (see [4]) that the symplectic graph $Sp(2\nu, q)$ is strongly regular with parameters

$$\left((q^{2\nu} - 1)/(q - 1), q^{2\nu-1}, q^{2\nu-2}(q - 1), q^{2\nu-2}(q - 1)\right)$$

and the chromatic number $q^\nu + 1$. For more results of the symplectic graph, the reader is referred to [1–4].

In this paper we shall study properties of the subconstituents $\Gamma_i([\alpha])$ for any $[\alpha] \in V(\Gamma)$, where $i = 1, 2$. Since $Sp_{2\nu}(\mathbb{F}_q)$ acts transitively on $V(Sp(2\nu, q))$ as automorphisms of $Sp(2\nu, q)$ (see [5] or [6]), in order to study subconstituents $\Gamma_i([\alpha])$, it suffices to consider $\Gamma_i([e_1])$ which is denoted by $\Gamma^{(i)}$ for simplicity.

This paper is organized as follows. In Section 2 we study some of the actions of $Sp_{2\nu}(\mathbb{F}_q)$ on the symplectic graph $Sp(2\nu, q)$ in preparation for later sections. In Section 3 we shall show that $\Gamma^{(1)}$ is not strongly regular, but a strictly Deza graph, and we determine the chromatic number of $\Gamma^{(1)}$. In Section 4 we shall show that $\Gamma^{(2)}$ is a strictly Deza graph except that $\nu = 2$, and $\Gamma^{(2)}$ is strongly regular when $\nu = 2$. Moreover, the chromatic number of $\Gamma^{(2)}$ is also determined.

2. Actions of $Sp_{2\nu}(\mathbb{F}_q)$ on $Sp(2\nu, q)$

In this section we study the actions of the symplectic group $Sp_{2\nu}(\mathbb{F}_q)$ on the symplectic graph $Sp(2\nu, q)$.

Let

$$G_1 = \{T \in Sp_{2\nu}(\mathbb{F}_q) \mid [e_1 T] = [e_1] \text{ and } [e_{\nu+1} T] = [e_{\nu+1}]\},$$
$$G_2 = \{T \in Sp_{2\nu}(\mathbb{F}_q) \mid [e_1 T] = [e_1] \text{ and } [e_2 T] = [e_2]\},$$

then G_1, G_2 are subgroups of $Sp_{2\nu}(\mathbb{F}_q)$. The following lemma is used in this section.

Lemma 2.1 (*See [5] or [6]*). *For any* $[\alpha], [\beta] \in V(Sp(2\nu, q))$, *we have the following:*

(i) *If* $[\alpha] \nsim [\beta]$, *then there exists a* $T \in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T] = [e_1]$ *and* $[\beta T] = [e_2]$.

(ii) *If* $[\alpha] \sim [\beta]$, *then there exists a* $T \in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T] = [e_1]$ *and* $[\beta T] = [e_{\nu+1}]$. ■

Based on Lemma 2.1, the actions of $Sp_{2\nu}(\mathbb{F}_q)$ on 3-element subsets of $V(Sp(2\nu, q))$ with various relationships are considered in Propositions 2.2–2.5.

Proposition 2.2. *Let* $[\alpha]$, $[\beta]$ *and* $[\gamma]$ *be three vertices of* $Sp(2\nu, q)$ *which are adjacent to each other, then there exists an element* $T \in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T] = [e_1]$, $[\beta T] = [e_{\nu+1}]$, *and* $[\gamma T]$ *is one of the following forms*

$$[e_1 + a_{\nu+1}e_{\nu+1}], \qquad [e_1 + e_2 + a_{\nu+1}e_{\nu+1}], \qquad [e_1 + a_{\nu+1}e_{\nu+1} + e_{\nu+2}], \tag{1}$$

where $a_{\nu+1} \in \mathbb{F}_q^*$.

Proof. By Lemma 2.1, there exists a $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $[\alpha T] = [e_1]$ and $[\beta T] = [e_{\nu+1}]$. Without loss of generality we can assume that $[\alpha] = [e_1]$, $[\beta] = [e_{\nu+1}]$, and $[\gamma] \in V(Sp(2\nu, q))$ satisfies $[\gamma] \sim [e_1]$ and $[\gamma] \sim [e_{\nu+1}]$. Write $[\gamma] = [a_1, \ldots, a_\nu, a_{\nu+1}, \ldots, a_{2\nu}]$. Since $[\gamma] \sim [e_1]$ and $[\gamma] \sim [e_{\nu+1}]$ we have $a_1 \neq 0$ and $a_{\nu+1} \neq 0$. Hence we can assume that $a_1 = 1$ and $[\gamma]$ is of the form $[\gamma] = [1, a_2, \ldots, a_\nu, a_{\nu+1}, \ldots, a_{2\nu}]$, where $a_{\nu+1} \in \mathbb{F}_q^*$. We distinguish the following two cases.

Case 1. $(a_{\nu+2}, \ldots, a_{2\nu}) = (0, \ldots, 0)$. If $(a_2, \ldots, a_\nu) = (0, \ldots, 0)$, then $[\gamma] = [e_1 + a_{\nu+1}e_{\nu+1}]$, the first vertex listed in (1). If $(a_2, \ldots, a_\nu) \neq (0, \ldots, 0)$, then there exists an $A \in GL_{\nu-1}(\mathbb{F}_q)$ such that $(a_2, \ldots, a_\nu)A = (1, 0, \ldots, 0)$. Take $T_1 = \mathrm{diag}(1, A, 1, {}^t(A^{-1}))$, then $T_1 \in G_1$ such that $[\gamma T_1] = [e_1 + e_2 + a_{\nu+1}e_{\nu+1}]$, the second vertex listed in (1).

Case 2. $(a_{\nu+2}, \ldots, a_{2\nu}) \neq (0, \ldots, 0)$. Then there exists a $B \in GL_{\nu-1}(\mathbb{F}_q)$ such that $(a_{\nu+2}, \ldots, a_{2\nu})B = (1, 0, \ldots, 0)$. Take $T_2 = \mathrm{diag}(1, {}^t(B^{-1}), 1, B)$, then $T_2 \in G_1$ and $[\gamma T_2]$ is of the form $[\gamma T_2] = [1, b_2, \ldots, b_\nu, a_{\nu+1}, 1, 0, \ldots, 0]$, where $(b_2, \ldots, b_\nu) = (a_2, \ldots, a_\nu)\,{}^t(B^{-1})$. Take

$$T_3 = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & -b_2 & \cdots & -b_\nu & & 1 & & \\ & \vdots & & & & & \ddots & \\ & -b_\nu & & & & & & 1 \end{pmatrix}, \tag{2}$$

then $T_3 \in G_1$ such that $[\gamma T_2 T_3] = [e_1 + a_{\nu+1}e_{\nu+1} + e_{\nu+2}]$, the third listed in (1). ■

Proposition 2.3. *Let* $[\alpha], [\beta], [\gamma] \in V(Sp(2\nu, q))$ *with* $[\alpha] \sim [\beta]$, $[\alpha] \sim [\gamma]$ *and* $[\beta] \nsim [\gamma]$, *then there exists a* $T \in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T] = [e_1]$, $[\beta T] = [e_{\nu+1}]$, *and* $[\gamma T]$ *is* $[e_2 + e_{\nu+1}]$ *or* $[e_{\nu+1} + e_{\nu+2}]$.

Proof. By Lemma 2.1 we can assume that $[\alpha] = [e_1]$, $[\beta] = [e_{\nu+1}]$, and $[\gamma] \in V(Sp(2\nu, q))$ satisfies $[\gamma] \sim [e_1]$ and $[\gamma] \nsim [e_{\nu+1}]$. Write $[\gamma] = [a_1, \ldots, a_\nu, a_{\nu+1}, \ldots, a_{2\nu}]$. From $[\gamma] \sim [e_1]$ and $[\gamma] \nsim [e_{\nu+1}]$ we deduce $a_1 = 0$ and $a_{\nu+1} \neq 0$. Hence $[\gamma]$ is of the form $[\gamma] = [0, a_2, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu}]$, where $(a_2, \ldots, a_\nu, a_{\nu+2}, \ldots, a_{2\nu}) \neq (0, \ldots, 0)$. We distinguish the following two cases.

Case 1. $(a_{\nu+2}, \ldots, a_{2\nu}) = (0, \ldots, 0)$. Then $(a_2, \ldots, a_\nu) \neq (0, \ldots, 0)$. There exists an $A \in GL_{\nu-1}(\mathbb{F}_q)$ such that $(a_2, \ldots, a_\nu)A = (1, 0, \ldots, 0)$. Take $T_1 = \mathrm{diag}(1, A, 1, {}^t(A^{-1}))$, then $T_1 \in G_1$ such that $[\gamma T_1] = [e_2 + e_{\nu+1}]$.

Case 2. $(a_{\nu+2}, \ldots, a_{2\nu}) \neq (0, \ldots, 0)$. Then there exists a $B \in GL_{\nu-1}(\mathbb{F}_q)$ such that $(a_{\nu+2}, \ldots, a_{2\nu})B = (1, 0, \ldots, 0)$. Take $T_2 = \mathrm{diag}(1, {}^t(B^{-1}), 1, B)$, then $T_2 \in G_1$ and $[\gamma T_2]$ is of the form $[\gamma T_2] = [0, b_2, \ldots, b_\nu, 1, 1, 0, \ldots, 0]$, where $(b_2, \ldots, b_\nu) = (a_2, \ldots, a_\nu)\,{}^t(B^{-1})$. Take T_3 as in (2), then $T_3 \in G_1$ and $[\gamma T_2 T_3] = [e_{\nu+1} + e_{\nu+2}]$. ∎

Proposition 2.4. *Let* $[\alpha], [\beta]$ *and* $[\gamma]$ *be three vertices of* $Sp(2\nu, q)$ *with* $[\alpha] \nsim [\beta]$, $[\alpha] \nsim [\gamma]$ *and* $[\beta] \sim [\gamma]$, *then there exists a* $T \in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T] = [e_1]$, $[\beta T] = [e_2]$ *and* $[\gamma T] = [e_{\nu+2}]$.

Proof. Since $[e_1] \nsim [e_2]$, by Lemma 2.1 we can assume that $[\alpha] = [e_1]$, $[\beta] = [e_2]$, and $[\gamma] \in V(Sp(2\nu, q))$ satisfies $[\gamma] \nsim [e_1]$ but $[\gamma] \sim [e_2]$. Write $[\gamma] = [a_1, a_2, \ldots, a_{2\nu}]$. From $[\gamma] \nsim [e_1]$ and $[\gamma] \sim [e_2]$ we obtain that $a_{\nu+1} = 0$ and $a_{\nu+2} \neq 0$. Hence we can assume $[\gamma] = [a_1, \ldots, a_\nu, 0, 1, a_{\nu+3}, \ldots, a_{2\nu}]$. We distinguish the following two cases.

Case 1. $(a_{\nu+3}, \ldots, a_{2\nu}) = (0, \ldots, 0)$. If $(a_3, \ldots, a_\nu) = (0, \ldots, 0)$, then $[\gamma] = [a_1 e_1 + a_2 e_2 + e_{\nu+2}]$. Take

$$T = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & I^{(\nu-2)} & & & \\ & -a_1 & & 1 & & \\ -a_1 & -a_2 & & & 1 & \\ & & & & & I^{(\nu-2)} \end{pmatrix}$$

then $T \in G_2$ such that $[\gamma T] = [e_{\nu+2}]$. If $(a_3, \ldots, a_\nu) \neq (0, \ldots, 0)$, then there exists an $A \in GL_{\nu-2}(\mathbb{F}_q)$ such that $(a_3, \ldots, a_\nu)A = (1, 0, \ldots, 0)$. Take

$$T_1 = \begin{pmatrix} I^{(2)} & & & \\ & A & & \\ & & I^{(2)} & \\ & & & {}^t(A^{-1}) \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ -a_1 & & 1 & & & & & \\ & & & I^{(\nu-3)} & & & & \\ & -a_1 & & & 1 & & a_1 & \\ & -a_2 & -1 & & & 1 & & \\ & -1 & & & & & 1 & \\ & & & & & & & I^{(\nu-3)} \end{pmatrix},$$

then $T_1 \in G_2$ such that $[\gamma T_1] = [e_{\nu+2}]$.

Case 2. $(a_{\nu+3}, \ldots, a_{2\nu}) \neq (0, \ldots, 0)$. Then there exists a $B \in GL_{\nu-2}(\mathbb{F}_q)$ such that $(a_{\nu+3}, \ldots, a_{2\nu})B = (1, 0, \ldots, 0)$. Let

$$T_2 = \begin{pmatrix} I^{(2)} & & & \\ & {}^t(B^{-1}) & & \\ & & I^{(2)} & \\ & & & B \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & 1 & 1 & & & & & \\ & & & I^{(\nu-3)} & & & & \\ & & & & 1 & & & \\ & & & & & 1 & -1 & \\ & & & & & & 1 & \\ & & & & & & & I^{(\nu-3)} \end{pmatrix},$$

then $T_2 \in G_2$ and $[\gamma T_2]$ is of the form $[\gamma T_3] = [b_1, \ldots, b_\nu, 0, 1, 0, \ldots, 0]$, which reduces to *Case* 1. ■

Proposition 2.5. *Let* $[\alpha], [\beta]$ *and* $[\gamma]$ *be three vertices of* $Sp(2\nu, q)$ *which are nonadjacent to each other, then there exists a* $T \in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T] = [e_1]$, $[\beta T] = [e_2]$, *and* $[\gamma T]$ *is one of the following forms*

$$[e_1 + e_2], \qquad [e_3], \qquad [e_{\nu+3}],$$

in which the latter two cases occur only when $\nu \geq 3$.

Proof. By Lemma 2.1 we can assume that $[\alpha] = [e_1]$, $[\beta] = [e_2]$ and $[\gamma] \in V(Sp(2\nu, q))$ satisfies $[\gamma] \nsim [e_1]$ and $[\gamma] \nsim [e_2]$. Write $[\gamma] = [a_1, \ldots, a_\nu, a_{\nu+1}, \ldots, a_{2\nu}]$. From $[\gamma] \nsim [e_1]$ and $[\gamma] \nsim [e_2]$ we know that $a_{\nu+1} = a_{\nu+2} = 0$. Hence we can write

$$[\gamma] = [a_1, \ldots, a_\nu, 0, 0, a_{\nu+3}, \ldots, a_{2\nu}],$$

where $(a_3, \ldots, a_\nu, a_{\nu+3}, \ldots, a_{2\nu}) \neq (0, \ldots, 0)$ or $a_1, a_2 \in \mathbb{F}_q^*$. We distinguish the following two cases.

Case 1. $(a_{\nu+3}, \ldots, a_{2\nu}) = (0, \ldots, 0)$. Then $(a_3, \ldots, a_\nu) = (0, \ldots, 0)$ but $a_1, a_2 \in \mathbb{F}_q^*$, or $(a_3, \ldots, a_\nu) \neq (0, \ldots, 0)$. If $(a_3, \ldots, a_\nu) = (0, \ldots, 0)$, then $[\gamma] = [a_1 e_1 + a_2 e_2]$ with $a_1, a_2 \in \mathbb{F}_q^*$. Take $T_1 = \mathrm{diag}(a_1^{-1}, a_2^{-1}, I^{(\nu-2)}, a_1, a_2, I^{(\nu-2)})$, then $T_1 \in G_2$ such that $[\gamma T_1] = [e_1 + e_2]$. If $(a_3, \ldots, a_\nu) \neq (0, \ldots, 0)$, then $\nu \geq 3$, and there exists an $A \in GL_{\nu-2}(\mathbb{F}_q)$ such that $(a_3, \ldots, a_\nu)A = (1, 0, \ldots, 0)$. Take

$$T_2 = \begin{pmatrix} I^{(2)} & & & \\ & A & & \\ & & I^{(2)} & \\ & & & {}^t(A^{-1}) \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ -a_1 & -a_2 & 1 & & & & & \\ & & & I^{(\nu-3)} & & & & \\ & & & & 1 & & a_1 & \\ & & & & & 1 & a_2 & \\ & & & & & & 1 & \\ & & & & & & & I^{(\nu-3)} \end{pmatrix}$$

then $T_2 \in G_2$ such that $[\gamma T_1] = [e_3]$.

Case 2. $(a_{\nu+3}, \ldots, a_{2\nu}) \neq (0, \ldots, 0)$. Then $\nu \geq 3$, and there exists a $B \in GL_{\nu-2}(\mathbb{F}_q)$ such that $(a_{\nu+3}, \ldots, a_{2\nu})B = (1, 0, \ldots, 0)$. Let $T_3 = \mathrm{diag}(I^{(2)}, {}^t(B^{-1}), I^{(2)}, B)$, then $T_3 \in G_2$ and $[\gamma T_3]$ is of the form

$$[\gamma T_3] = [a_1, a_2, b_3, \ldots, b_\nu, 0, 0, 1, 0, \ldots, 0],$$

where $(b_3, \ldots, b_\nu) = (a_3, \ldots, a_\nu)\,{}^t(B^{-1})$. If $(b_4, \ldots, b_\nu) = (0, \ldots, 0)$, take

$$T_4 = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & I^{(\nu-3)} & & & & \\ & & -a_1 & & 1 & & & \\ & & -a_2 & & & 1 & & \\ -a_1 & -a_2 & -b_3 & & & & 1 & \\ & & & & & & & I^{(\nu-3)} \end{pmatrix},$$

then $T_4 \in G_2$ such that $[\gamma T_3 T_4] = [e_{\nu+3}]$. If $(b_4, \dots, b_\nu) \neq (0, \dots, 0)$, then there exists a $C \in GL_{\nu-3}(\mathbb{F}_q)$ such that $(b_4, \dots, b_\nu)C = (1, 0, \dots, 0)$. Let

$$T_5 = \begin{pmatrix} I^{(3)} & & & \\ & C & & \\ & & I^{(3)} & \\ & & & {}^t(C^{-1}) \end{pmatrix} \begin{pmatrix} I^{(2)} & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & I^{(\nu-4)} & & & & \\ & & & & I^{(2)} & & & \\ & 0 & -1 & & & 1 & & \\ & -1 & 0 & & & & 1 & \\ & & & & & & & I^{(\nu-4)} \end{pmatrix},$$

then $T_5 \in G_2$ and $[\gamma T_3 T_5] = [a_1 e_1 + a_2 e_2 + b_3 e_3 + e_{\nu+3}]$, which reduces to the case just treated. ■

3. The subconstituent $\Gamma^{(1)}$

Now we study properties of the subconstituent $\Gamma^{(1)}$. For any vertex $[\alpha] = [a_1, \dots, a_{2\nu}]$ of $\Gamma^{(1)}$, from $e_1 K {}^t\alpha \neq 0$ we obtain $a_{\nu+1} \neq 0$. So any vertex $[\alpha]$ of $\Gamma^{(1)}$ has a unique matrix representation of the form

$$[a_1, \dots, a_\nu, 1, a_{\nu+2}, \dots, a_{2\nu}], \tag{3}$$

where $a_1, \dots, a_\nu, a_{\nu+2}, \dots, a_{2\nu} \in \mathbb{F}_q$.

Before showing that $\Gamma^{(1)}$ is not strongly regular, we first prove the following two propositions.

Proposition 3.1. *Let $[\alpha_1]$ and $[\alpha_2]$ be any two vertices of $\Gamma^{(1)}$ satisfying $[\alpha_1] \sim [\alpha_2]$. Then $|\Gamma^{(1)}([\alpha_1]) \cap \Gamma^{(1)}([\alpha_2])| = (q-2)q^{2\nu-2}$ or $(q-1)^2 q^{2\nu-3}$.*

Proof. Note that $[e_{\nu+1}]$, $[e_1 + x e_{\nu+1}]$, $[e_1 + e_2 + x e_{\nu+1}]$ and $[e_1 + x e_{\nu+1} + e_{\nu+2}]$ are vertices of $\Gamma^{(1)}$ for any $x \in \mathbb{F}_q^*$, and

$$[e_{\nu+1}] \sim [e_1 + x e_{\nu+1}], \quad [e_{\nu+1}] \sim [e_1 + e_2 + x e_{\nu+1}], \quad [e_{\nu+1}] \sim [e_1 + x e_{\nu+1} + e_{\nu+2}].$$

For any fixed $x \in \mathbb{F}_q^*$, let

$$\mathcal{M}_1 = \{[\alpha] \in V\Gamma^{(1)} | [\alpha] \sim [e_{\nu+1}] \text{ and } [\alpha] \sim [e_1 + x e_{\nu+1}]\},$$
$$\mathcal{M}_2 = \{[\alpha] \in V\Gamma^{(1)} | [\alpha] \sim [e_{\nu+1}] \text{ and } [\alpha] \sim [e_1 + e_2 + x e_{\nu+1}]\},$$
$$\mathcal{M}_3 = \{[\alpha] \in V\Gamma^{(1)} | [\alpha] \sim [e_{\nu+1}] \text{ and } [\alpha] \sim [e_1 + x e_{\nu+1} + e_{\nu+2}]\}.$$

In order to prove the lemma, it suffices by Proposition 2.2 to show that

$$\{|\mathcal{M}_1|, |\mathcal{M}_2|, |\mathcal{M}_3|\} = \{(q-2)q^{2\nu-2}, (q-1)^2 q^{2\nu-3}\}.$$

Let $[\alpha] \in \mathcal{M}_1$ be of the form (3). From $[\alpha] \sim [e_{\nu+1}]$ and $[\alpha] \sim [e_1 + e_{\nu+1}]$ we deduce that $a_1 \neq 0$ and $1 - x a_1 \neq 0$, i.e., $a_1 \neq 0$ and $a_1 \neq x^{-1}$. So

$$|\mathcal{M}_1| = (q-2)q^{2\nu-2}.$$

Let $[\alpha] \in \mathcal{M}_2$ be of the form (3). From $[\alpha] \sim [e_{\nu+1}]$ and $[\alpha] \sim [e_1 + e_2 + x e_{\nu+1}]$ we deduce that $a_1 \neq 0$ and $1 - x a_1 + a_{\nu+2} \neq 0$. The number of $(a_1, a_2) \in \mathbb{F}_q^{(2)}$ satisfying $a_1 \neq 0$ is $(q-1)q$, and the number of $(a_1, a_2) \in \mathbb{F}_q^{(2)}$ satisfying both $a_1 \neq 0$ and $1 - x a_1 - a_2 = 0$ is $q - 1$. So the number of $(a_1, a_2) \in \mathbb{F}_q^{(2)}$ satisfying both $a_1 \neq 0$ and $1 - xa_1 - a_2 \neq 0$ is $(q-1)q - (q-1) = (q-1)^2$. Thus we have

$$|\mathcal{M}_2| = (q-1)^2 q^{2\nu-3}.$$

Similar to the computation of $|\mathcal{M}_2|$ we can deduce that $|\mathcal{M}_3| = (q-1)^2 q^{2\nu-3}$. ■

Proposition 3.2. *Let* $[\alpha_1]$ *and* $[\alpha_2]$ *be any two vertices of* $\Gamma^{(1)}$ *such that* $[\alpha_1] \nsim [\alpha_2]$. *Then* $|\Gamma^{(1)}([\alpha_1]) \cap \Gamma^{(1)}([\alpha_2])| = (q-1)^2 q^{2\nu-3}$.

Proof. Let

$$\mathcal{M}_1 = \{[\alpha] \in V\Gamma^{(1)} \mid [\alpha] \sim [e_{\nu+1}] \text{ and } [\alpha] \sim [e_2 + e_{\nu+1}]\},$$
$$\mathcal{M}_2 = \{[\alpha] \in V\Gamma^{(1)} \mid [\alpha] \sim [e_{\nu+1}] \text{ and } [\alpha] \sim [e_{\nu+1} + e_{\nu+2}]\}.$$

Note that $[e_{\nu+1}]$, $[e_2 + e_{\nu+1}]$ and $[e_{\nu+1} + e_{\nu+2}]$ are vertices of $\Gamma^{(1)}$, and

$$[e_{\nu+1}] \nsim [e_2 + e_{\nu+1}], \qquad [e_{\nu+1}] \nsim [e_{\nu+1} + e_{\nu+2}].$$

To prove the lemma, it suffices by Proposition 2.3 to show that $|\mathcal{M}_1| = |\mathcal{M}_2| = (q-1)^2 q^{2\nu-3}$. Let $[\alpha] \in \mathcal{M}_1$ be of the form (3). Then $a_1 \neq 0$ and $a_{\nu+2} - a_1 \neq 0$. Clearly, the number of $[\alpha] \in V(\Gamma^{(1)})$ with $a_1 \neq 0$, $a_{\nu+2} - a_1 \neq 0$ is $(q-1)^2 q^{2\nu-3}$, i.e., $|\mathcal{M}_1| = (q-1)^2 q^{2\nu-3}$. Similarly we can deduce that $|\mathcal{M}_2| = (q-1)^2 q^{2\nu-3}$. ■

A graph G is an (n, k, b, a)-*Deza graph* if $|V(G)| = n$ and for any $x, y \in V(G)$

$$|G(x) \cap G(y)| = \begin{cases} a \text{ or } b & \text{if } x \neq y, \\ k & \text{if } x = y, \end{cases}$$

where n, k, b and a are integers such that $0 \leq a \leq b \leq k \leq n$. Clearly, strongly regular graphs are Deza graphs. The only difference between a strongly regular graph and a Deza graph is that the size $|G(x) \cap G(y)|$ does not necessarily depend on whether $x \sim y$. These graphs are called *Deza graphs* because they were introduced (in a slightly more restricted form) by Antoine and Michel Deza [7]. A *strictly Deza graph* is a Deza graph that is not strongly regular and has two diameters (see [8]).

The following theorem shows that the subconstituent $\Gamma^{(1)}$ of $Sp(2\nu, q)$ is not strongly regular. More precisely, we have

Theorem 3.3. *The subconstituent* $\Gamma^{(1)}$ *is a strictly Deza graph with parameters*

$$\left(q^{2\nu-1}, (q-1)q^{2\nu-2}, (q-1)^2 q^{2\nu-3}, (q-2)q^{2\nu-2}\right). \tag{4}$$

Proof. It is clear from (3) that $|V(\Gamma^{(1)})| = q^{2\nu-1}$. In order to prove that $\Gamma^{(1)}$ is a strictly Deza graph with parameters as (4), it suffices by Propositions 3.1 and 3.2 to show that $\Gamma^{(1)}$ is regular with valency $(q-1)q^{2\nu-2}$. By Lemma 2.1 it suffices to count the number k of $[\alpha] \in V(\Gamma^{(1)})$ satisfying $[\alpha] \sim [e_{\nu+1}]$. Let $[\alpha]$ be of the form (3). From $[\alpha] \sim [e_{\nu+1}]$ we obtain $a_1 \neq 0$. Therefore $k = (q-1)q^{2\nu-2}$, the valency of $\Gamma^{(1)}$. ■

In order to determine the chromatic number $\chi(\Gamma^{(1)})$, Lemma 3.4 is recalled, an upper bound is given in Theorem 3.5, then we show by virtue of Proposition 3.6 that $\chi(\Gamma^{(1)}) = q^\nu$ in Theorem 3.7.

Lemma 3.4 (*See [4]*). *The graph $Sp(2\nu, q)$ is $(q^\nu + 1)$-partite. That is, there exists subsets $Y_1, Y_2, \ldots, Y_{q^\nu+1}$ of $V(Sp(2\nu, q))$ such that*

$$V(Sp(2\nu, q)) = Y_1 \cup Y_2 \cup \cdots \cup Y_{q^\nu+1},$$

where $Y_i \cap Y_j = \emptyset$ for all $i \neq j$, and there is no edge of $Sp(2\nu, q)$ joining two vertices of the same subset. Moreover, the subsets $Y_1, \ldots, Y_{q^\nu+1}$ can be chosen so that for any two distinct indices i and j, every $\alpha \in Y_i$ is adjacent with exactly $q^{\nu-1}$ vertices of Y_j. ■

Theorem 3.5. *The graph $\Gamma^{(1)}$ is q^ν-partite. That is, there exist subsets $X_1, X_2, \ldots, X_{q^\nu}$ of $V(\Gamma^{(1)})$ such that*

$$V(\Gamma^{(1)}) = X_1 \cup X_2 \cup \cdots \cup X_{q^\nu},$$

where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $\Gamma^{(1)}$ joining two vertices of the same subset. Moreover, every X_i contains exactly $q^{\nu-1}$ vertices of $\Gamma^{(1)}$.

Proof. By Lemma 3.4 we can write that

$$V(Sp(2\nu, q)) = Y_1 \cup Y_2 \cup \cdots \cup Y_{q^\nu+1},$$

where $Y_i \cap Y_j = \emptyset$ for all $i \neq j$, and there is no edge of $Sp(2\nu, q)$ joining two vertices of the same Y_i. Hence we have

$$V(\Gamma^{(1)}) = (V(\Gamma^{(1)}) \cap Y_1) \cup (V(\Gamma^{(1)}) \cap Y_2) \cup \cdots \cup (V(\Gamma^{(1)}) \cap Y_{q^\nu+1}).$$

Let $X_i = V(\Gamma^{(1)}) \cap Y_i$, $i = 1, 2, \ldots, q^\nu + 1$. Then

$$V(\Gamma^{(1)}) = X_1 \cup X_2 \cup \cdots \cup X_{q^\nu} \cup X_{q^\nu+1},$$

where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $\Gamma^{(1)}$ joining two vertices of the same X_i. Since $[e_1] \in V(Sp(2\nu, q))$, $[e_1] \in Y_i$ for some i $(1 \leq i \leq q^\nu + 1)$. Without loss of generality we can assume $[e_1] \in Y_{q^\nu+1}$. So $X_{q^\nu+1} = V(\Gamma^{(1)}) \cap Y_{q^\nu+1} = \emptyset$. Thus

$$V(\Gamma^{(1)}) = X_1 \cup X_2 \cup \cdots \cup X_{q^\nu}.$$

By Lemma 3.4, $[e_1] \in Y_{q^\nu+1}$ is adjacent with exactly $q^{\nu-1}$ vertices in Y_i for each $i (1 \leq i \leq q^\nu)$. So $|X_i| = |V(\Gamma^{(1)}) \cap Y_i| = q^{\nu-1}$. ■

Proposition 3.6. *Let W be any maximal totally isotropic subspace in the 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, and α be a vector of $\mathbb{F}_q^{(2\nu)}$ but $\alpha \notin W$. Then the number of 1-dimensional subspaces $[\beta]$ of W satisfying $\alpha K^t \beta \neq 0$ is $q^{\nu-1}$.*

Proof. Since $Sp_{2\nu}(\mathbb{F}_q)$ acts transitively on each set of totally isotropic subspaces of the same dimension (see [6, Theorem 3.7]), without loss of generality we can assume that $W = [e_1, e_2, \ldots, e_\nu]$. Write $\alpha = (a_1, a_2, \ldots, a_{2\nu})$. It follows from $\alpha \notin W$ that $(a_{\nu+1}, \ldots, a_{2\nu}) \neq (0, \ldots, 0)$. Let

$$\beta = x_1 e_1 + x_2 e_2 + \cdots + x_\nu e_\nu, \quad x_i \in \mathbb{F}_q, \quad 1 \leq i \leq \nu.$$

Clearly, the number of $\beta \in W$ satisfying $\alpha K^t \beta = 0$ is $q^{\nu-1}$. Hence the number of $\beta \in W$ satisfying $\alpha K^t \beta \neq 0$ is $q^\nu - q^{\nu-1}$, and the number of 1-dimensional subspaces $[\beta]$ of W satisfying $\alpha K^t \beta \neq 0$ is $(q^\nu - q^{\nu-1})/(q - 1) = q^{\nu-1}$. ■

Theorem 3.7. $\chi(\Gamma^{(1)}) = q^\nu$.

Proof. It follows from Theorem 3.5 that $\chi(\Gamma^{(1)}) \leq q^{\nu}$. Suppose that $\Gamma^{(1)}$ is n-partite. Then there exist subsets $X_1, X_2, \ldots, X_n$ of $V(\Gamma^{(1)})$ such that

$$V(\Gamma^{(1)}) = X_1 \cup X_2 \cup \cdots \cup X_n, \tag{5}$$

where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $\Gamma^{(1)}$ joining two vertices of the same X_i for all $1 \leq i \leq n$. We shall show that $n \geq q^{\nu}$, hence $\chi(\Gamma^{(1)}) = q^{\nu}$. Suppose that $n < q^{\nu}$. From (5) we know that

$$\sum_{i=1}^{n} |X_i| = |V(\Gamma^{(1)})| = q^{2\nu-1} = q^{\nu} \cdot q^{\nu-1} > n \cdot q^{\nu-1}.$$

Hence there exists some X_i such that $|X_i| > q^{\nu-1}$. Let W_i be the subspace generated by all α such that $[\alpha] \in X_i$. Then W_i is totally isotropic and dim $W_i \geq \nu$, where dim W_i is the dimension of W_i. So dim $W_i = \nu$, i.e., W_i is maximally totally isotropic. Note that $e_1 K^t \beta \neq 0$ for all $[\beta] \in X_i$, i.e., there exist at least $|X_i|$ (which is greater than $q^{\nu-1}$ in number) 1-dimensional subspaces $[\beta]$ of W_i such that $e_1 K^t \beta \neq 0$, which is contrary to Proposition 3.6. ■

Note that $\chi(\Gamma) = q^{\nu} + 1$, from Theorem 3.7 we can obtain the relation between the chromatic number of Γ and that of $\Gamma^{(1)}$, namely

Corollary 3.8. $\chi(\Gamma) = \chi(\Gamma^{(1)}) + 1$. ■

4. The subconstituent $\Gamma^{(2)}$

In this section we study properties of the subconstituent $\Gamma^{(2)}$. Let $[\alpha] = [a_1, \ldots, a_{2\nu}]$ be any vertex of $\Gamma^{(2)}$. From $e_1 K^t \alpha = 0$ we obtain $a_{\nu+1} = 0$. So any vertex $[\alpha]$ of $\Gamma^{(2)}$ is of the form

$$[\alpha] = [a_1, \ldots, a_{\nu}, 0, a_{\nu+2}, \ldots, a_{2\nu}], \tag{6}$$

where $a_2, \ldots, a_{\nu}, a_{\nu+2}, \ldots, a_{2\nu} \in \mathbb{F}_q$ are not all zero, and we have

$$|V(\Gamma^{(2)})| = (q^{2\nu-1} - q)/(q-1). \tag{7}$$

Proposition 4.1. *Let* $[\alpha_1]$ *and* $[\alpha_2]$ *be any two vertices of* $\Gamma^{(2)}$ *such that* $[\alpha_1] \sim [\alpha_2]$. *Then* $|\Gamma^{(2)}([\alpha_1]) \cap \Gamma^{(2)}([\alpha_2])| = (q-1)q^{2\nu-3}$.

Proof. Let

$$\mathcal{M} = \{[\alpha] \in V\Gamma^{(2)} | [\alpha] \sim [e_2] \text{ and } [\alpha] \sim [e_{\nu+2}]\}.$$

Note that $[e_2], [e_{\nu+2}] \in V(\Gamma^{(2)})$ and $[e_2] \sim [e_{\nu+2}]$. To prove the proposition it suffices by Proposition 2.4 to show $|M| = (q-1)q^{2\nu-3}$. Let $[\alpha] \in \mathcal{M}$ be of the form (6). From $[\alpha] \sim [e_2]$ and $[\alpha] \sim [e_{\nu+2}]$ we deduce that $a_2, a_{\nu+2} \in \mathbb{F}_q^*$. So $|\mathcal{M}| = (q-1)q^{2\nu-3}$. ■

Proposition 4.2. *Let* $[\alpha_1]$ *and* $[\alpha_2]$ *be any two vertices of* $\Gamma^{(2)}$ *satisfying* $[\alpha_1] \nsim [\alpha_2]$. *Then*

$$|\Gamma^{(2)}([\alpha_1]) \cap \Gamma^{(2)}([\alpha_2])| = \begin{cases} q^2 & \text{if } \nu = 2, \\ q^{2\nu-2} \text{ or } (q-1)q^{2\nu-3} & \text{if } \nu \geq 3. \end{cases}$$

Proof. Consider first the case of $\nu \geq 3$. Let

$$\mathcal{M}_1 = \{[\alpha] \in V\Gamma^{(2)} \mid [\alpha] \sim [e_2] \text{ and } [\alpha] \sim [e_1 + e_2]\},$$
$$\mathcal{M}_2 = \{[\alpha] \in V\Gamma^{(2)} \mid [\alpha] \sim [e_2] \text{ and } [\alpha] \sim [e_3]\},$$
$$\mathcal{M}_3 = \{[\alpha] \in V\Gamma^{(2)} \mid [\alpha] \sim [e_2] \text{ and } [\alpha] \sim [e_{\nu+3}]\}.$$

To prove the proposition, it suffices by Proposition 2.5 to show that

$$\{|\mathcal{M}_1|, |\mathcal{M}_2|, |\mathcal{M}_3|\} = \{q^{2\nu-2}, (q-1)q^{2\nu-3}\}.$$

Let $[\alpha] \in \mathcal{M}_1$ be of the form (6). From $[\alpha] \sim [e_2]$ and $[\alpha] \sim [e_1 + e_2]$ we obtain $a_{\nu+2} \neq 0$. Hence we have

$$|\mathcal{M}_1| = q^{2\nu-2}.$$

Let $[\alpha] \in \mathcal{M}_2$ be of the form (6). From $[\alpha] \sim [e_2]$ and $[\alpha] \sim [e_3]$ we deduce that $a_{\nu+2} \neq 0$ and $a_{\nu+3} \neq 0$. It follows that

$$|\mathcal{M}_2| = (q-1)q^{2\nu-3}.$$

Similar to the computation of $|\mathcal{M}_2|$ we can obtain $|\mathcal{M}_3| = (q-1)q^{2\nu-3}$.

For the case of $\nu = 2$, by Proposition 2.5 we only need to compute $|\mathcal{M}_1|$. Then any element of $\mathcal{M}_1$ is of the form $[a_1, a_2, 0, 1]$, where $a_1, a_2 \in \mathbb{F}_q$. So $|\mathcal{M}_1| = q^2$. ■

Theorem 4.3. *The subconstituent $\Gamma^{(2)}$ is a strongly regular graph with parameters*

$$(q(q+1), q^2, q(q-1), q^2)$$

when $\nu = 2$, and is a strictly Deza graph with parameters

$$\left((q^{2\nu-1} - q)/(q-1), q^{2\nu-2}, q^{2\nu-2}, (q-1)q^{2\nu-3}\right)$$

when $\nu \geq 3$.

Proof. $|V(\Gamma^{(2)})| = (q^{2\nu-1} - q)/(q-1)$ is given by (7). By Propositions 4.1 and 4.2 we know that $\Gamma^{(2)}$ is strongly regular with parameters $(q(q+1), q^2, q(q-1), q^2)$ when $\nu = 2$. Now consider the case of $\nu \geq 3$. It is enough by Propositions 4.1 and 4.2 to show that $\Gamma^{(2)}$ is regular with valency $q^{2\nu-2}$. By Lemma 2.1 it suffices to count the number k of $[\alpha] \in V(\Gamma^{(2)})$ satisfying $[\alpha] \sim [e_2]$. Let $[\alpha] \in V(\Gamma^{(2)})$ be of the form (6). From $[\alpha] \sim [e_2]$ we obtain $a_{\nu+2} \neq 0$. So $k = q^{2\nu-2}$, the valency of $\Gamma^{(2)}$. ■

Note that the complement of the strongly regular graph considered in Theorem 4.3 is strongly regular with parameters $(q(q+1),\ q-1,\ q-2,\ 0)$, which consists of $q+1$ copies of the clique K_q. Similarly to Theorem 3.5 we have

Theorem 4.4. *The subconstituent $\Gamma^{(2)}$ is $(q^\nu + 1)$-partite. That is, there exist subsets $X_1, X_2, \ldots, X_{q^\nu+1}$ of $V(\Gamma^{(2)})$ such that*

$$V(\Gamma^{(2)}) = X_1 \cup X_2 \cup \cdots \cup X_{q^\nu+1},$$

where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $\Gamma^{(2)}$ joining two vertices of the same X_i. Moreover, among $X_1, X_2, \ldots, X_{q^\nu+1}$ there is exactly one X_i such that $|X_i| = (q^\nu - q)/(q-1)$, and $|X_j| = (q^{\nu-1} - 1)/(q-1)$ for all $1 \leq j \leq q^\nu + 1$ but $j \neq i$.

Proof. By Lemma 3.4 we can write

$$V(Sp(2\nu, q)) = Y_1 \cup Y_2 \cup \cdots \cup Y_{q^\nu+1},$$

where $Y_i \cap Y_j = \emptyset$ for all $i \neq j$, and there is no edge of $Sp(2\nu, q)$ joining two vertices of the same Y_i. Hence we have

$$V(\Gamma^{(2)}) = (V(\Gamma^{(2)}) \cap Y_1) \cup (V(\Gamma^{(2)}) \cap Y_2) \cup \cdots \cup (V(\Gamma^{(2)}) \cap Y_{q^\nu+1}).$$

Let $X_i = V(\Gamma^{(2)}) \cap Y_i$, $i = 1, 2, \ldots, q^\nu + 1$. Then

$$V(\Gamma^{(2)}) = X_1 \cup X_2 \cup \cdots \cup X_{q^\nu+1},$$

where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $\Gamma^{(2)}$ joining two vertices of the same X_i. Since $[e_1] \in V(Sp(2\nu, q))$, $[e_1] \in Y_i$ for some i. Without loss of generality we can assume that $[e_1] \in Y_{q^\nu+1}$. Note that there is no edge of $Sp(2\nu, q)$ joining two vertices of $Y_{q^\nu+1}$; we have

$$Y_{q^\nu+1} - \{[e_1]\} \subseteq V(\Gamma^{(2)}).$$

So

$$X_{q^\nu+1} = V(\Gamma^{(2)}) \cap Y_{q^\nu+1} = Y_{q^\nu+1} - \{[e_1]\}$$

and

$$|X_{q^\nu+1}| = \frac{q^\nu - 1}{q - 1} - 1 = \frac{q^\nu - q}{q - 1}.$$

By Lemma 3.4, $[e_1]$ is adjacent to exactly $q^{\nu-1}$ vertices in Y_i for each i $(1 \leq i \leq q^\nu)$ and $|Y_i| = (q^\nu - 1)/(q - 1)$. Note that for each $1 \leq i \leq q^\nu$,

$$X_i = \{[\alpha] \in Y_i | [\alpha] \nsim [e_1]\},$$

we have

$$|X_i| = \frac{q^\nu - 1}{q - 1} - q^{\nu-1} = \frac{q^{\nu-1} - 1}{q - 1}, \quad 1 \leq i \leq q^\nu. \quad \blacksquare$$

Let G, G' be two graphs. The *lexicographic product* $G[G']$ of G and G' is a graph with the vertex set $V(G) \times V(G')$ and with the adjacency defined by

$$(u_1, u_2) \sim (v_1, v_2) \quad \text{if and only if} \quad u_1 \sim v_1, \text{ or } u_1 = v_1 \text{ but } u_2 \sim v_2$$

for any $u_1, v_1 \in V(G)$ and $u_2, v_2 \in V(G')$. Note that $\mathbb{F}_q$ is a clique with $\mathbb{F}_q$ as its vertex set and the adjacency defined by $x \sim y$ if and only if $x \neq y$. Then the complement $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ is a coclique. We have

Theorem 4.5. *The subconstituent $\Gamma^{(2)}$ is isomorphic to $\overline{\mathbb{F}}_q[Sp(2\nu - 2, q)]$.*

Proof. For the sake of definiteness we assume that each vertex $[\alpha]$ of $\Gamma^{(2)}$, which is of the form (6), is written such that the first nonzero element among $a_2, \ldots, a_\nu, a_{\nu+2}, \ldots, a_{2\nu}$ is chosen as 1. It is evident that the mapping

$$\begin{array}{cccc} f: & \Gamma^{(2)} & \longrightarrow & \overline{\mathbb{F}}_q[Sp(2\nu-2, q)] \\ & \left[a_1 e_1 + \sum_{i=2}^{\nu}(a_i e_i + a_{\nu+i} e_{\nu+i})\right] & \longmapsto & \left(a_1, \left[\sum_{i=2}^{\nu}(a_i e_i + a_{\nu+i} e_{\nu+i})\right]\right) \end{array}$$

is an isomorphism between the graphs $\Gamma^{(2)}$ and $\overline{\mathbb{F}}_q[Sp(2\nu-2,q)]$. ■

Clearly, $\chi(\overline{\mathbb{F}}_q[Sp(2\nu-2,q)]) = \chi(Sp(2\nu-2,q)) = q^{\nu-1}+1$. From Theorem 4.5 we have

Theorem 4.6. $\chi(\Gamma^{(2)}) = q^{\nu-1}+1$. ■

From Theorem 4.6 we can obtain the relation between the chromatic number of Γ and that of $\Gamma^{(2)}$, namely

Corollary 4.7. $\chi(\Gamma) - 1 = q(\chi(\Gamma^{(2)}) - 1)$. ■

By virtue of Theorem 4.6 we may obtain an interesting result of the 2ν-dimensional symplectic space over $\mathbb{F}_q$.

Theorem 4.8. *Let $[\alpha]$ be any 1-dimensional subspace of the symplectic space $\mathbb{F}_q^{(2\nu)}$. Then there exist precisely $q^{\nu-1}+1$ maximally totally isotropic subspaces $V_1, V_2, \ldots, V_{q^{\nu-1}+1}$ of $\mathbb{F}_q^{(2\nu)}$ such that $V_i \cap V_j = [\alpha]$ for all $i \neq j$, $1 \leq i, j \leq q^{\nu-1}+1$.*

Proof. Since $Sp_{2\nu}(\mathbb{F}_q)$ acts transitively on $V(Sp(2\nu,q))$ (see [5] or [6]), we can assume that $[\alpha] = [e_1]$. By Theorem 4.6, there exist subsets $X_1, X_2, \ldots, X_{q^{\nu-1}+1}$ of $V(\Gamma^{(2)})$ such that

$$V(\Gamma^{(2)}) = X_1 \cup X_2 \cup \cdots \cup X_{q^{\nu-1}+1},$$

where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $\Gamma^{(2)}$ joining two vertices of the same X_i. Note that $[e_1] \notin X_i$ and for any $[x] \in [X_i]$ we have $e_1 K^t x = 0$. So $|X_i| \leq \frac{q^\nu-1}{q-1} - 1 = \frac{q^\nu-q}{q-1}$. From

$$\frac{q^\nu - q}{q-1} \cdot (q^{\nu-1}+1) = \frac{q^{2\nu-1}-q}{q-1} = |V(\Gamma^{(2)})| = \sum_{i=1}^{q^{\nu-1}+1} |X_i|$$

we obtain

$$|X_i| = \frac{q^\nu - q}{q-1}, \quad i = 1, 2, \ldots, q^{\nu-1}+1.$$

Let $Y_i = X_i \cup \{[e_1]\}$ and V_i be the subspace of $\mathbb{F}_q^{(2\nu)}$ generated by all α such that $[\alpha] \in Y_i$. Then V_i is maximal totally isotropic, and $V_i \cap V_j = [e_1]$ for all $i \neq j$, $1 \leq i, j \leq q^{\nu-1}+1$. ■

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